15th June, '26

Introduction to periodically driven quantum systems

— by Sayan Sir (HRI)

What is periodically driven sys$^{\underline{m}}$? Understanding NEq. Sys.

→ $\boxed{H(t) = H_0 + g(t)\,V + g^*(t)\,V^\dagger}$ → Floquet Hamiltonian

$$g(t+T) = g(t)$$

This Hamiltonian has periodic time dependence.

Imagine observing the system ~~stroboscopically~~ at $\underbrace{t = 0, T, 2T, \ldots, nT}_{\text{stroboscopic time}}$. stroboscopically

"Stroboscopic" dynamics is governed by a time-independent " " U, where, $U_F = \mathcal{T}\left[\exp\left(-i\int_0^T dt\, H(t)\right)\right]$ → time ordered unitary op. (Floquet op.)

This U_F is the time ordered ~~un~~ Floquet unitary operator.

- **Relevant parameters:** i) Amplitude of the drive
  ii) Frequency of the drive
  iii) Phase of the drive wrt starting time.

[~~Ang~~ g(T) is controlling the period, g(T) → square wave. So, we only have t = 0 → T/2 and T/2 → T. So, only 2 freq. As we include more & more incommensurate freq(s), the drive Gives ~~us~~ most generic non eq^{lm} systems

✳ g(T) can be from any distribution ⇒ RMT ⇒ Gaussian ⇒ Any ...]

[Generic non equ^{m} sys^{m} can be achieved by adding more & more incommensurate freq.].

- **Goals of the Field:**

1) Designing effective floquet Hamiltonian.
2) Intrinsically non-equilibrium Phases of Matter.

   Example:- Time crystal (Forbidden in equ$^{\underline{m}}$), Floquet Anomalous TI.

3) Evading Floquet Heating

- Non-local Hamiltonian
- Floquet Anamolous Topological Insulator (filling is never well defined but we have a dynamical signature) ≡ Photonic sys.

Q: If I have a PD Hamiltonian, how do I solve SE. for this system?

SE for Floquet sys$^{\underline{m}}$: $i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$

P.T.O...

$$i\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$\Rightarrow \quad i\partial_t u(t) = H(t)\, u(t) \quad \longrightarrow (\text{ODE})$$

with $u(0) = \mathbb{1}$.

Any solⁿ of $u(t)$ is called the fundamental matrix

Note: $i\partial_t u(t+T) = H(t+T)\, u(t+T)$ — $T \equiv$ Time period.

$i\partial_t u(t+T) = H(t)\, u(t+T)$

Both $u(t)$ and $u(t+T)$ are fundamental matrices.

$$\boxed{u(t+T) = u(t)\, u_F}$$

↳ (Theorem of ODE) Check.

$u_F \rightarrow$ Time independent.

So, $u(t+T, 0) = u(t+T, T)\, u(T, 0)$

similarly $u(t+T, T) = u(t, 0)$

* Def. of u_F : $\boxed{u_F = u(T, 0)}$

[$u(3T+\alpha, 0) = u(3T+\alpha, 3T)\, u(3T, 2T)\, u(2T, T)\, u(T, 0)$

$t = 2T + \alpha$ where α runs from 0 to T ; $0 < \alpha < T$]

There exists a non-singular continuously differentiable "Matrix-valued" function P which satisfies : with a period T

$P(t+T) = P(t)$, such that $u(t) = P(t)\, e^{-itH_F}$

↑ Floquet Hamiltonian time independent

Set, $P(t) = u(t)\, e^{itH_F}$

$$P(t+T) = u(t+T)\, e^{i(t+T)H_F}$$
$$= u(t)\, e^{itH_F}\, e^{iTH_F}\, u_F$$
$$= u(t)\, e^{itH_F} \quad = P(t)$$

So, $\boxed{u_F = e^{-iTH_F}}$ $\longrightarrow$ here it should be defined like this.

and $P(0) = \mathbb{1}$.

In stroboscopic times: $\boxed{u(nT) = e^{-inTH_F} = (u_F)^N}$

$P(t)$'s are micro-motion operators that actually tells us what happens in the middle

There exists a rotating frame i.e. related with the lab frame by the condⁿ: $\underbrace{u(t)}_{\text{lab}} = P(t)\, \underbrace{e^{-itH_F}}_{\text{rot}}\, P^{\dagger}(0) \Rightarrow \boxed{u(t) = P(t)\, e^{-itH_F}}$

$$P^{-1} = P^{\dagger}$$

$$\begin{tikzcd} \text{Lab} \arrow[r, bend left] & \text{Rotating} \arrow[l, bend left, "P"] \end{tikzcd}$$

We can go from one frame to another by this transformation.

If $P(t) \to$ known then H_F can be calculated. (approximately)

At $t=0$, both frames co-incides. As time evolves the wavefn's follows certain relns.

$$|\psi(t)\rangle_{lab} = P(t)\, U_{rot}(t,0)\, |\psi(0)\rangle$$

$$i\partial_t |\psi(t)\rangle_{lab} = i\partial_t P(t) |\psi(t)\rangle_{rot}$$

$$= i\dot{P} |\psi(t)\rangle_{rot} + P(t) H_{rot}(t) |\psi(t)\rangle_{rot}$$

Now, $H_{lab} = i\dot{P}P^{-1} + P H_{rot} P^{-1}$

$H_{rot} = P^{\dagger} H_{lab} P - \underbrace{iP^{\dagger}\dot{P}}$ — coriolis type term so, it is called rotation.

"drive" → non linear

• Magnus expansion

Simple Example : 1. Two-Level System (Rabii Oscillations).

$$H_{lab} = \begin{bmatrix} B_z & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & -B_z \end{bmatrix}$$

Q: What is the expression of $P(t)$?

$$\Rightarrow P(t) = e^{-i\omega t\, \sigma_z/2}.$$

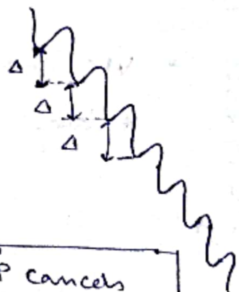
Using $P(t)$ we'll get :-

$H_{rot} = (B_z - \omega/2)\sigma^z + \gamma\sigma^x$. and we get Rabii Oscillation with freq. $\Omega = \sqrt{(B_z - \omega/2)^2 + \gamma^2}$ → eigen value of the H_{rot}.

Now: $B_z |1\rangle\langle 1| - B_z |0\rangle\langle 0| + \gamma e^{i\omega t} |1\rangle\langle 0| + \gamma e^{-i\omega t} |0\rangle\langle 1|$

2. A particle hopping on a lattice in presence of $\vec{E}$.

$$H_{lab} = H = \sum_j -J(a_{j+1}^{\dagger} a_j + h.c.) + \sum_j j\Delta n_j$$



$\Delta \to$ tilt ($\Delta\uparrow$, localization $\uparrow$) → w/o Int.

→ Bloch oscillations in QM due to group velocities.

$$P(t) = \exp\left[-it\Delta \sum_j j n_j\right]$$

$\boxed{iP^{\dagger}\dot{P} \text{ cancels the dipole term}}$

$$H_{rot}(t) = -\sum_j J(e^{it\Delta} a_{j+1}^{\dagger} a_j + h.c)$$

$\boxed{\text{Period} = \frac{2\pi}{\Delta}}$

Wannier & Zener.

micromotions.

Now making the tilt time-dependent $\Delta(t) = \cancel{A\omega\cos\omega t}$ $A\omega\cos\omega t$

Our Hamiltonian becomes :-

$$H = \sum_j -J\,(a_{j+1}^{\dagger} a_j + h.c.) + \underbrace{A\omega \cos\omega t \sum_j j\, n_j}_{H_2 \to \text{problematic term.}}$$

$$P(t) = e^{-iA\sin\omega t \sum_j j\, n_j}$$

$$H_{rot} = -J \sum e^{iA\sin\omega t}\, a_{j+1}^{\dagger} a_j + h.c. \quad \text{and}$$

$$\int_0^T dT/_T\; e^{-iA\sin\omega t} = J_0(A) \to \text{Bessel funct}^n.$$

[If $A\uparrow$ or $\omega\uparrow$, then, $H_2 \to$ not perturbation. But in the rotating frame, I can take the high freq. limit.

So, $P(t) \to$ approximate is giving me an approximate H_F ~~that~~ could ~~tell~~ me much more informations ($A\uparrow$, $\omega\uparrow$ than the lab frame).]

Sys $\to$ Drive tuned $\downarrow$ tune prop of sys.

- Dynamical localization (coherent .....)
- inverse freq. expansion

~~Whatever I do in the rot frame.~~ (Frame independent)

* Stroboscopic dynamics does not change from frame to frame.

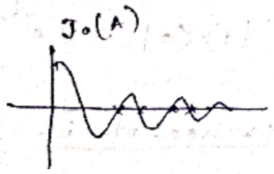
* Micro-motion can change frame to frame.

* In rot. frame, floquet H can be calculated in a more controllable way. in large ω expansion.

* Micromotion is not that imp. quantity.

* As stroboscopic dynamics do not change, so physics is not changing from frame to frame.

[$J_0(A)$



$\to$ zeros of Bessel's $f^n \Rightarrow \cancel{t\omega}\; H_F = 0$?? talk to peers not sure

* We have system $\to$ we introduce drive $\to$ tune that drive $\to$ tune the properties of the system.

- Kapitza Pendulum. $\to$ Another example.
- Airy limit

* H_F is same in both frames, ~~H_{F}~~

$$H_F = \frac{1}{T}\int dT\, H_{rot}(t) = J\, J_0(A) \left[\sum_i a_i^{\dagger} a_{i+1} + h.c\right].$$

* This transformation help us to find H_F in a more ~~contro~~ controllable manner.

→ Applications :- 1) Frustrated magnetism
2) Gauge field.

Lecture : 2

Date :- 17th June,' 2026

Recall : Stroboscopic evolution : is governed by $u_F = e^{-iH_FT}$

so find the Hamiltonian, $H_F = i/_T \ln u_F$
↳ not well defined
→ diff in diff branch cuts.

The exp of u_F in terms of its eigenstates :-

$$u_F = \sum_n e^{-i\theta_n} |n\rangle\langle n|$$ → $\hbar\omega$ quanta of energy sys^{m} can absorb or emit it. (quasi-energy)

Q: How do I compute H_F?

→ Inverse Freq. expansion, i.e. (IFE) $H(t) = \sum_m H_m e^{im\omega t}$

Expand this $H_F = \sum_n \Omega_n$ ...

$\boxed{\text{* Effective } H_F \text{ would be Hermitian}}$

• IFE : Magnus Expansion :

$\Omega_1 = H_0$ → averaging the Hamiltonian

$\Omega_2 = \sum_{m_i} [H_{m_i} - H_{-m}] / 2m\omega$

and Ω_3, ~~are~~ and Ω_n are much more complicated.

1) Floquet Fermi's Golden Rule ⇒ Heating.

2) u_F belongs to certain RMT circular ensemble.

3) Floquet Prethermalization.

Q: How a transport distinguish b/w pre therm to therm? • Anomalous diffusion

1)* Floquet Fermi Golden Rule : [I'keda & Polkovnikov, PRB 104, 134308 (2021)]

$$\underset{\substack{\uparrow\\ \text{exact Floquet Hamiltonian}}}{H^F_{ex}} = \underbrace{H^F}_{\substack{\text{obtained by IFE} \\ \text{(approximate)}}} + \underbrace{V}_{\text{correction.}}$$

[H_F → has a gauge dep. To avoid this Van-Vleck expansion is needed]

We will work in the high frequency ~~limit~~ regime.

$$H(t) = H_F + \underbrace{g(t) V}_{\text{perturbation}} \quad \text{where} \quad g(t+T) = g(t)$$

$\uparrow$ periodically driven term.

Q: What is the transition rates b/w different eigenstates of H_F?

A: Now, $\delta u_F = u_F^\dagger u \neq 1$ $\quad u \to$ full Time evol^n operator

$u_F = e^{-iH_F T}$

$|n\rangle \equiv$ eigenstates of u_F, i.e H_F

We assume that: $\underset{\text{density matrix}}{\rho(t)} = \sum_n P_n(t) |n\rangle\langle n|$

$$\boxed{\begin{aligned} H_F|n\rangle &= E_n|n\rangle \\ u_F|n\rangle &= e^{-i\theta_n}|n\rangle \end{aligned}}$$

How, does this $P_n(t)$ change?

$$\frac{dP_n}{dt} = \sum_m \left[\omega_{m\to n} P_m(t) - \omega_{n\to m} P_n(t) \right] \quad \boxed{\Rightarrow \text{Exactly ETH}}$$

Claim: $\quad \omega_{m\to n} = \Omega \sum_{l \in \mathbb{Z}} \delta(\theta_n - \theta_m - 2\pi l) \, |\langle n|\delta u_F|m\rangle|^2$

when $m \neq n$

$l \to$ phase of $|m\rangle$ $\cdots$ $|n\rangle$ (eigenstates).

* Steady state sol^n: $\boxed{\frac{dP_n}{dt} = 0} \implies P_m(t) = P_n(t)$

$\boxed{\text{So, } P_n = \text{const } \forall n}$ $\to$ Attractor of this equ^n.

So, density matrix becomes the identity matrix

$\longrightarrow$ which will lead to infinite ~~time~~ dim. sys^m

Overall time evolution operator: u

Now, $u = u_F u_F^\dagger u = u_F \delta u_F \simeq u_F (\mathbb{1} + i\delta\hat{k})$

At long times:-

$$P_{m\to n} = |\langle n| u^N |m\rangle|^2 \quad [N \gg 1]$$

Transition prob. of ~~at~~ $|m\rangle$ to $|n\rangle$ at time NT.

$$\simeq \left| \sum_{k=1}^{N} \langle n| u_F^k \, \delta\hat{k} \, u_F^{N-k} |m\rangle \right|^2$$

$$\simeq \left| \sum_{k=1}^{N} e^{-i(\theta_n - \theta_m)k} \right|^2 |\langle n|\delta u|m\rangle|^2$$

(P.T.O)

$$\simeq 2\pi N \, |\langle n| \delta u | m\rangle|^2 \sum_{l} \delta(\theta_n - \theta_m - 2\pi l)$$

$$\left[\langle n| u_F + i u_F \delta\hat{k} |m\rangle = e^{-i\theta_n} \delta_{mn} + i e^{-i\theta_n} \langle n| \delta\hat{k} |m\rangle \right]$$

2) *Time Evolution Operator of Floquet Unitary

Consider a Floquet Hamiltonian :

$$\hat{H}(\tau) = \hat{H}_{ave} + f(\tau)\, \hat{A}$$

Define : $\hat{u}(\tau) \,|\phi_n(\tau)\rangle = e^{-i\theta_n(\tau)} |\phi_n(\tau)\rangle$.

$|\phi_n(\tau)\rangle \rightarrow$ instantaneous eigenstates of the Floquet Hamiltonian when $\tau = 1$.

$$\boxed{\frac{\partial}{\partial t} \hat{u} = \frac{1}{i} H(t)\, u(t)} \longrightarrow \text{SE}.$$

Use the fact : $\frac{\partial}{\partial \tau} \langle \phi_m(\tau) | u^\dagger(\tau) u(\tau) | \phi_n(\tau)\rangle = 0$

as we know: $\langle \phi_m(\tau) | u^\dagger(\tau)\, u(\tau) | \phi_n(\tau)\rangle = \delta_{mn}$.

Now : $\frac{\partial}{\partial \tau} \langle \phi_m(\tau) | u^\dagger(\tau)\, u(\tau) | \phi_n(\tau)\rangle = 0$

$$\Rightarrow \frac{i}{T} \left[1 - e^{i[\theta_n(\tau) - \theta_m(\tau)]} \right] \langle \phi_n(\tau) | \frac{\partial}{\partial \tau} | \phi_m(\tau)\rangle$$
$$= \langle \phi_n(\tau) | \hat{H}(\tau) | \phi_m(\tau)\rangle.$$

set, $\tau = 1$, generically R.H.S $\neq 0$, $\theta_n(\tau) \neq \theta_m(\tau)$

As they (θ's) are in circles, so, eigenvalues repulsion occurs.

$\boxed{\text{Eigenvalue repulsion on a circle}}$

$\hookrightarrow$ Circular Ensemble instead of Gaussian ensemble.

Since, u_F is drawn from the circular ensemble, these eigenstates are Haar - random.

- We want to compute $\langle H_{ave}\rangle$ at long times (0 to NT where $N \gg 1$) :-

$$\langle H_{ave} \rangle (t = NT) = \langle \psi_0 | (u_F^\dagger)^N \, H_{ave} \, (u_F)^N | \psi_0 \rangle.$$

(P.T.O)

for large N :-

$$\langle H_{ave} \rangle \Big|_{t=NT} \simeq \sum_n |\langle \psi_0 | \phi_n \rangle|^2 \langle \phi_n | H_{ave} | \phi_n \rangle .$$

Now, ϕ_n's are Haar-random vectors. so,

$$\langle H_{ave} \rangle \Big|_{t=NT.} = \frac{1}{\text{Dim}[H_{ave}]} \, \text{Tr}[H_{ave}]$$

$\longrightarrow$ This is the manifestation of infinite tempera-ture state.

3)* Floquet Prethermalization :

Lab frame $\xrightarrow{e^{ikt}}$ Exact Rotating frame

$H_0 + V(t)$ $\qquad\qquad$ H_F (static)

(Non-local quantity) in terms of k-bodyness

$\xrightarrow{e^{ik_n t}}$ Local H

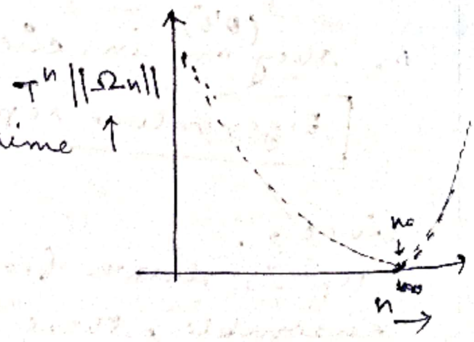
[Fermi-Golden rule suppressed] $\longrightarrow$ $H = \underbrace{H_F^{(n)}}_{\text{local}} + \underbrace{V_n(t)}_{\text{perturbation}}$

$\equiv$ dynamics of the sys

Ref : Kuwahara, Mori and Saito [Annals of Physics, 367, 96 (2016)]

$$H_F = \sum_{n=0}^{\infty} T^n \Omega_n \longrightarrow \text{(Floquet - Magnus)}$$

This series is asymptotic series.



This optimal order "n_0" sets a time - scale, i.e.

$$e^{-iH_F T} \simeq e^{-iH_F(n_0) T}.$$