First-passage percolation along lattice cylinders

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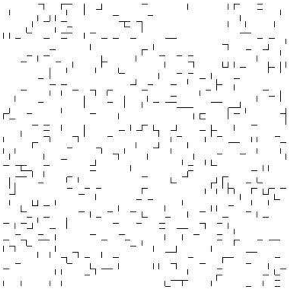

Figure: $p = 0.1$

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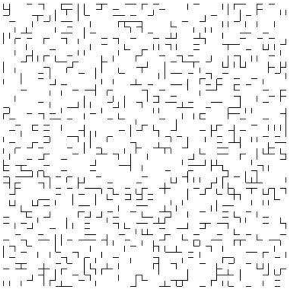

Figure: $p = 0.2$

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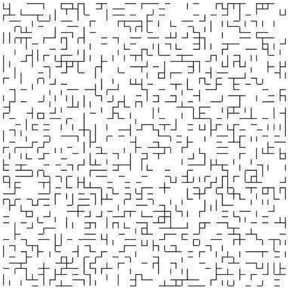

Figure: $p = 0.3$

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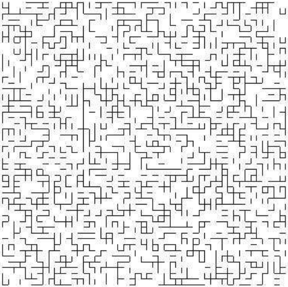

Figure: $p = 0.4$

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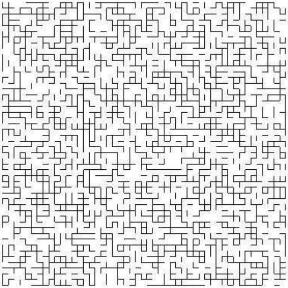

Figure: $p = 0.5$

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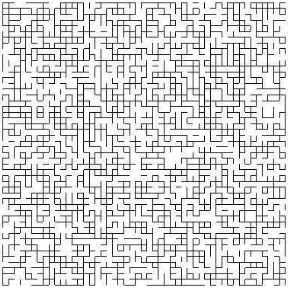

Figure: $p = 0.6$

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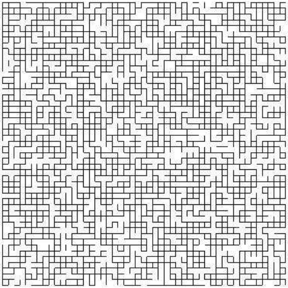

Figure: $p = 0.7$

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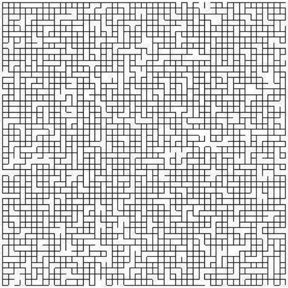

Figure: $p = 0.8$

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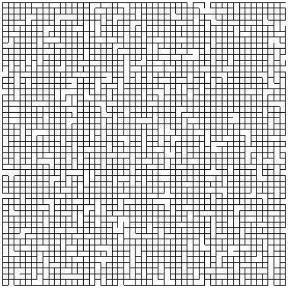

Figure: $p = 0.9$

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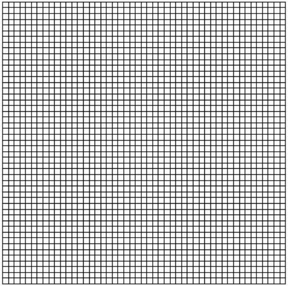

Figure: $p = 1.0$

Phase Transition

Phase transition at $p_c(2) = 1/2$ on $\mathbb{Z}^2$:

- if $p < 1/2$: no infinite cluster almost surely. (sub-critical)
- if $p > 1/2$: one unique infinite cluster almost surely. (super-critical)
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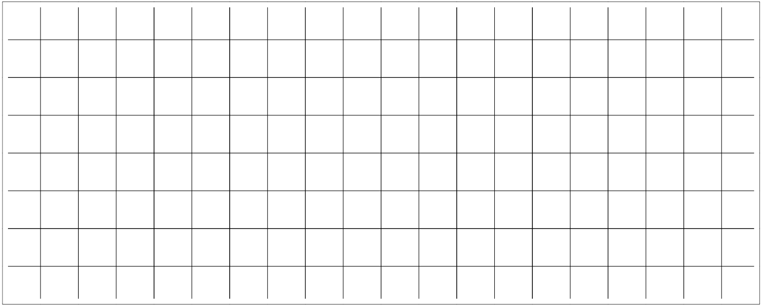
For every dimension d, $p_c(d)$ exists and is strictly in between 0 and 1.

First-passage pecolation



- Now consider the same lattice model where each edge e has independent and identically distributed (i.i.d.) random nonnegative weight ω_e from a fixed distribution F.

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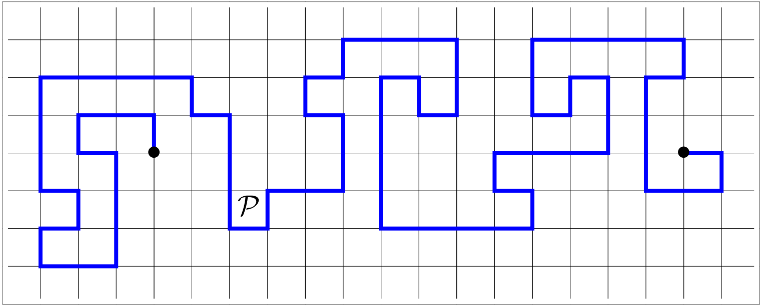

- For any path $\mathcal{P}$, define the passage time for $\mathcal{P}$ by

$$\omega(\mathcal{P}) := \sum_{e \in \mathcal{P}} \omega_e.$$

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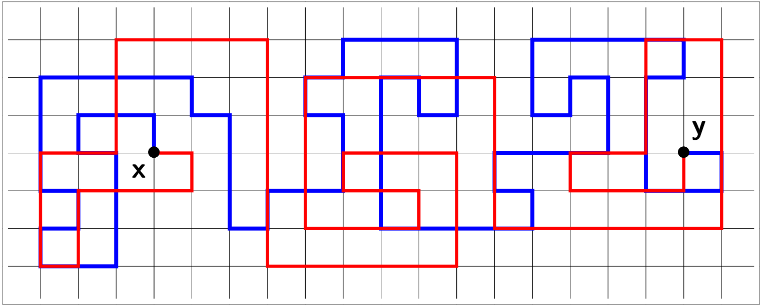

- For two vertices $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, the first-passage time $a(\mathbf{x}, \mathbf{y})$ is defined as the minimum passage time over all paths from $\mathbf{x}$ to $\mathbf{y}$.

Model

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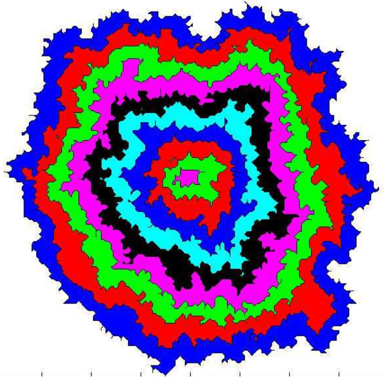

Figure: Cluster growth

Known results: Mean behavior

- When the edge weights have finite mean, by subadditivity

$$\nu(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n} a(\mathbf{0}, \lfloor n\mathbf{x} \rfloor)$$

exists and is finite for all $\mathbf{x} \in \mathbb{R}^d$ (HW'65).

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Subadditivity: Let a_n be a sequence of real numbers that satisfies $a_{n+m} \leq a_n + a_m$ for all n, m. Then $\lim_{n\to\infty} a_n/n$ exists and equals $\inf_{n\geq 1} a_n/n$.

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Proof: For any fixed integer $m \geq 1$ we have

$$\frac{a_n}{n} \leq \frac{km}{km+r} \cdot \frac{a_m}{m} + \frac{a_r}{n}$$

where $n = km + r$ and $0 \leq r < m$. Letting $n \to \infty$ we have

$$\limsup \frac{a_n}{n} \leq \inf \frac{a_m}{m} \leq \liminf \frac{a_n}{n}.$$

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- A shape theorem was proved by Cox and Durrett('81) which says that for every small $\varepsilon > 0$ and large enough t

$$(1+\varepsilon)B \subseteq \frac{B_t}{t} \subseteq (1+\varepsilon)B$$

a.s. where $B_t = \{\mathbf{x} : a(\mathbf{0}, \lfloor \mathbf{x} \rfloor) \leq t\}$ and $B = \{\mathbf{x} : \nu(\mathbf{x}) \leq 1\}$.

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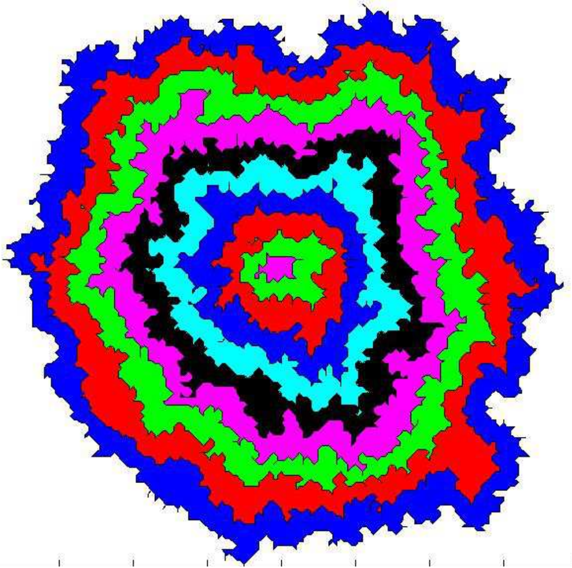
- Kesten('86) proved that, $\nu(\mathbf{x}) > 0$ iff $\mathbb{P}(\omega_e = 0) < p_c(d)$ where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$.

Mean behavior



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- Bounds on $\mathbb{V}\mathrm{ar}(a(\mathbf{0}, n\mathbf{x}))$ when $\mathbb{P}(\omega_e = 0) < p_c(d)$:
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- Nothing is known about the limiting distribution of $a(\mathbf{0}, n\mathbf{x})$ when $\mathbb{P}(\omega_e = 0) < p_c(d)$.

Predictions

- $\mathbb{V}\mathrm{ar}(a(\mathbf{0}, n\mathbf{x})) \approx n^{2\chi}$ and the minimizing path deviates from the straight line path joining $\mathbf{0}$ to $n\mathbf{x}$ by at most n^{ξ} where χ, ξ depends only on d.

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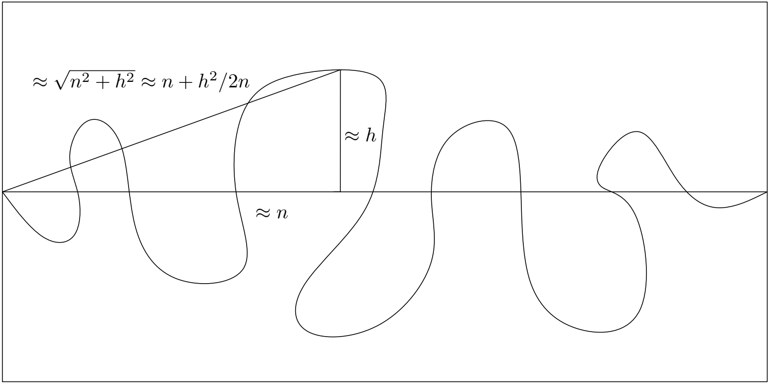
- For $d = 2$, it is predicted that, $\chi = 1/3$ and $\xi = 2/3$. *i.e.*,

$$\mathbb{V}\mathrm{ar}(a(\mathbf{0}, n\mathbf{x})) \approx n^{2/3}$$

and the minimizing path is in $\mathbb{Z} \times [-n^{2/3+\varepsilon}, n^{2/3+\varepsilon}]$ for any $\varepsilon > 0$ according to the predictions.

KPZ heuristics



$$\frac{n^{2\xi}}{2n} \approx n^{\chi} \implies 2\xi - 1 = \chi.$$

Our result

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Theorem (Chatterjee and D. (2010))

Suppose $\mathbb{P}(\omega_e = 0) < p_c(d)$ *with all moments finite. Let* h_n *be a sequence of integers satisfying* $h_n \ll n^{1/(d+1)}$*. Then we have*

$$\frac{a_n(h_n) - \mathbb{E}[a_n(h_n)]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}} \Longrightarrow \textit{Standard Gaussian as } n \to \infty.$$

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In particular, in two-dimension, we have Gaussian Limit as long as $h_n \ll n^{1/3}$.

Moment bounds

- We have,
$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[a_n(h_n)] = \nu(1,0,\ldots,0)$$
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- For all n, h_n, we have

$$\mathbb{E}[|a_n(h_n) - \mathbb{E}[a_n(h_n)]|^k] \le cn^{k/2}$$

where c depends only on F and d.

Fixed h case

Assume $h_n = h$ for all n for fixed $h \in (0, \infty)$

- Both

$$\mu(h) := \lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}[a_n(h)] \text{ and } \sigma^2(h) := \lim_{n\to\infty} \frac{1}{n}\,\mathbb{V}\mathrm{ar}(a_n(h))$$

  exist and are positive for any non-degenerate distribution F on $[0, \infty)$, but their values depend on h, F.

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exist and are positive for any non-degenerate distribution F on $[0, \infty)$, but their values depend on h, F.

- The scaled process $\{(n\sigma^2(h))^{-1/2}(X(nt) - nt\mu(h))\}_{t\geq 0}$ converges in distribution to the standard Brownian motion as $n \to \infty$ where

$$X(n) = a_n(h) \text{ for } n \geq 1$$

and extended by linear interpolation.

Take home message

Below the height threshold the first-passage time has Gaussian fluctuation and Gaussianity breaks down at the height threshold.

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- Any L^2 function f of gaussian variables can be expanded in Hermite orthonormal basis as $f = \sum_{k=0}^{\infty} \sum_{||\mathbf{m}||_1 = k} c_{\mathbf{m}} \tilde{H}_{\mathbf{m}}$ and by Parseval's identity we have

$$\mathbb{V}\mathrm{ar}(f) = \sum_{k \geq 1} a_k^2$$

where $a_k^2 = \sum_{||\mathbf{m}||_1 = k} c_{\mathbf{m}}^2$. Moreover we also have,

$$\mathbb{E}[|\nabla f|^2] = \sum_{k \geq 1} k a_k^2$$

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- Hence for $h_n \to \infty$, $a_n(h_n)$ is noise sensitive, which implies that any constant level Fourier mass is negligible compared to the variance.
- The hardest part is to analyse the full spectrum.

Simulation studies (Variance)

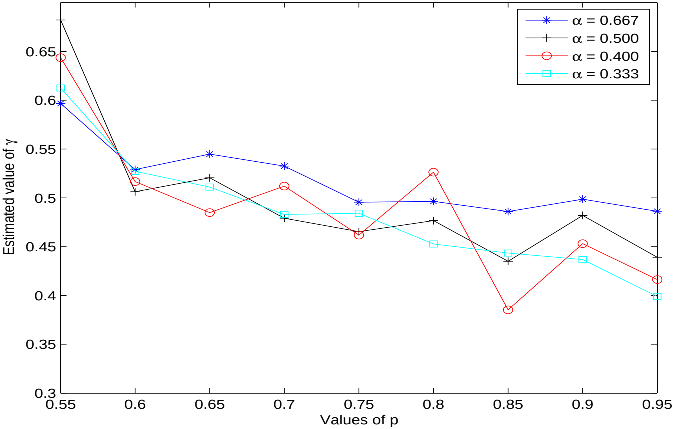


Figure: Plot of estimated value of γ vs. p for different values of α under i.i.d. Bernoulli(p) data and the assumption that $\mathbb{V}\mathrm{ar}(a_n(n^\alpha)) = O(nh_n^{-\gamma})$.

Simulation studies (CLT)

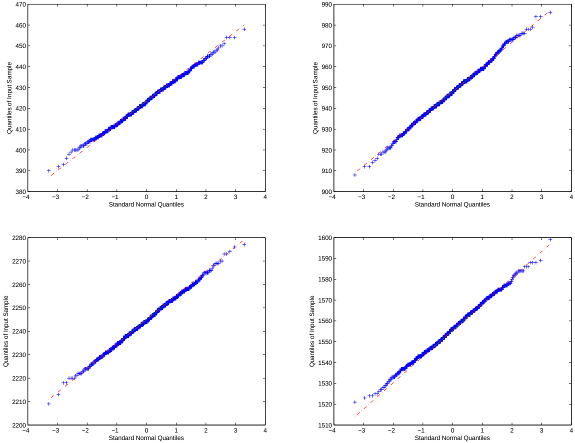
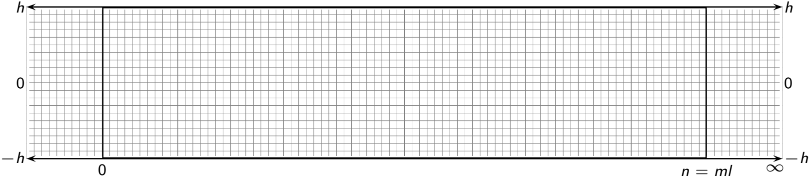


Figure: QQ plots based on simulation data for $a_n(n^{1/2})$ for $n = 3000$ against normal distribution for Bernoulli(p) edge weights, $p = 0.6, 0.7, 0.8, 0.9$ in clockwise direction starting from top left.

Sketch of the proof: $d = 2$ and $h_n \ll n^\alpha$



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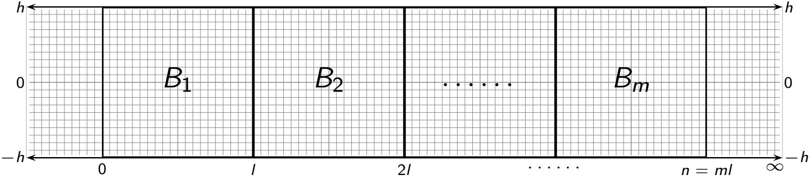

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$$B_i = [(i-1)l, il] \times [-h_n, h_n] \text{ for } 1 \leq i \leq m.$$

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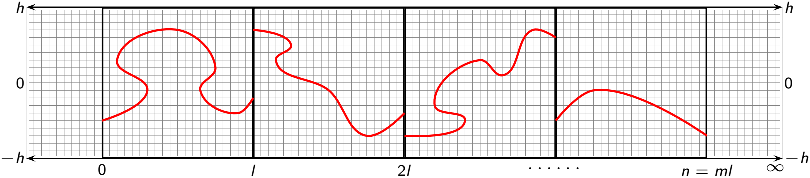

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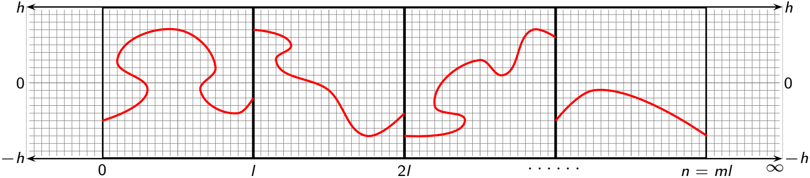

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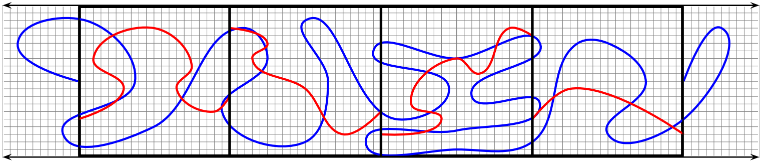
- X_i's are independent for $1 \le i \le m$ with $X_i \stackrel{\mathrm{d}}{=} T(l, h_n)$ where

$$T(l, h_n) := \inf\{\omega(\mathcal{P}) : \mathcal{P} \text{ is a path joining left and right boundaries of } [0, l] \times [-h_n, h_n]\}.$$

Approximation as i.i.d. sum



- We have

$$a_n(h_n) \geq X_1 + X_2 + \cdots X_m.$$

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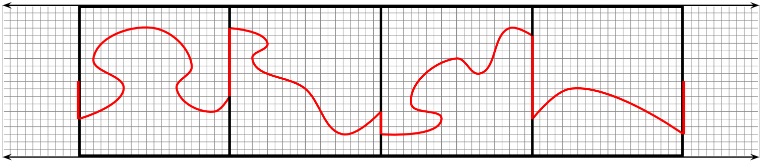

- We have

$$a_n(h_n) \geq X_1 + X_2 + \cdots X_m.$$

- We also have

$$a_n(h_n) \leq X_1 + X_2 + \cdots X_m + Z$$

where Z is sum of all edge-weights in the left/right boundaries of B_i.

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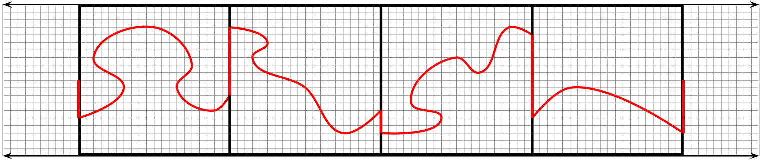

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- $\mathbb{E}\left|\frac{a_n(h_n) - \mathbb{E}[a_n(h_n)]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}} - \sum_{i=1}^{m} \frac{X_i - \mathbb{E}[X_i]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}}\right|^2 \leq \frac{4\mathbb{E}[Z^2]}{\mathbb{V}\mathrm{ar}(a_n(h_n))}.$

Approximation as i.i.d. sum (contd.)

- Now $\mathbb{E}[Z^2] = O((mh_n)^2)$ and $\mathbb{V}\mathrm{ar}(a_n(h_n)) = \Omega(n/h_n)$.

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- Now $\mathbb{E}[Z^2] = O((mh_n)^2)$ and $\mathbb{V}\mathrm{ar}(a_n(h_n)) = \Omega(n/h_n)$.

- Thus $a_n(h_n)$ is approximately a sum of i.i.d. random variables when

$$\mathbb{E}[Z^2] \approx (mh_n)^2 \ll n/h_n \le \mathbb{V}\mathrm{ar}(a_n(h_n))$$

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- Using Lyapounov's condition, CLT holds for $X_1 + \cdots + X_m$ when

$$\frac{m\,\mathbb{E}\,|X_1 - \mathbb{E}[X_1]|^k}{(m\,\mathbb{V}\mathrm{ar}(X_1))^{k/2}} = o(1).$$

Central Limit Theorem upto $n^{1/5}$ ($h_n \ll n^{\alpha}, m \approx n^{1-\beta}$)

- Using moment upper bound we have

$$\frac{m\,\mathbb{E}\,|X_1 - \mathbb{E}[X_1]|^k}{(m\,\mathbb{V}\mathrm{ar}(X_1))^{k/2}} \leq const \times \frac{m \times l^{k/2}}{\left(\frac{ml}{h_n}\right)^{k/2}}$$

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- Taking $\beta = 4/5$ and k large we have CLT for $a_n(h_n)$ when $h_n \ll n^{1/5}$.

Additive moment bound

- Note that when $l = m_2 l_2$ we have

$$\mathbb{E}\,|T(l,h_n) - \sum_{i=1}^{m_2} T_i(l_2,h_n)|^k = O((m_2 h_n)^k)$$

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- Moreover, by Rosenthal's inequality we have, for i.i.d. mean zero random variables $X_1, \ldots, X_m$, $k \geq 2$,

$$\mathbb{E}[|X_1 + \cdots + X_m|^k] \leq C_k \max\{m^{k/2} \cdot \mathbb{E}[X^2]^{k/2}, m \cdot \mathbb{E}[|X|^k]\}.$$

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- The previous bound was using the fact that

$$\mathbb{E}[|X_1 + \cdots + X_m|^k] \leq C_k' \; m^{k/2} \cdot \mathbb{E}[|X|^k].$$

Renormalization

- Each X_i has the same properties as the variable $a_n(h_n)$. Using this self-similarity upto two level with $l \approx n^{7/8}$, $l_2 \approx n^{3/4}$ one can prove CLT for $h_n = o(n^\alpha)$ with $\alpha < \frac{1}{4}$.
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- Thus taking t large enough, we have CLT for $h_n = o(n^\alpha)$ with $\alpha < 1/3$.

Why should CLT hold upto $n^{2/3}$ in 2-dimension

- Let's look at at the error terms in the proof of CLT. Recall that $n = ml$ with $l \approx n^{\beta}, h_n = o(n^{\alpha})$ and each sub-block has size $l \times h_n$.

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\begin{aligned}
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- In our proof, we used $\gamma = 1$ and vertical fluctuation h_n.

Ongoing work

- For $d = 2$, prove that

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- Other works: long-range first-passage percolation with multiple phase transition. Invariant measures for nonlinear Schrödinger equation.

Thank you!