Non-perturbative Renormalization of Lattice QCD

Part IV/V

Stefan Sint

Trinity College Dublin & NIC@DESY-Zeuthen

Saha Institute of Nuclear Physics

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Contents

- The coupling in the Schrödinger functional scheme
- SF schemes for composite operators (→ mass renormalization)
- Step scaling functions and how to get them from lattice approximants
- Some results by the ALPHA collaboration
- Symmetries and Ward identities
- Wilson quarks and chiral Ward identities
Definition of the SF coupling [Lüscher et al. ’92]

- Choose abelian and spatially constant boundary gauge fields:
  
  \[ C_k = \frac{i}{L} \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}, \quad C'_k = \frac{i}{L} \begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix}, \quad k = 1, 2, 3 \]

- with angles taken to be linear functions of a parameter \( \eta \):
  
  \[ \phi_1 = \eta - \frac{\pi}{3}, \quad \phi'_1 = -\phi_1 - \frac{4\pi}{3}, \]
  \[ \phi_2 = -\frac{1}{2} \eta, \quad \phi'_2 = -\phi_3 + \frac{2\pi}{3}, \]
  \[ \phi_3 = -\frac{1}{2} \eta + \frac{\pi}{3}, \quad \phi'_3 = -\phi_2 + \frac{2\pi}{3}. \]

- The gauge action has an absolute minimum for:
  
  \[ B_0 = 0, \quad B_k = \left[ x_0 C'_k + (L - x_0) C_k \right] / L, \quad k = 1, 2, 3. \]

  i.e. other gauge fields with the same action must be gauge equivalent to \( B_\mu \)
Definition of the SF coupling

- Define the effective action of the induced background field

\[ \Gamma[B] = - \ln \mathcal{Z}[C, C'] \]

- In perturbation theory the effective action has the expansion

\[ \Gamma[B] \sim g_0^{-2} \Gamma_0[B] + \Gamma_1[B] + O(g_0^2) \]

- Definition of the SF coupling:

\[ \bar{g}^2(L) = \frac{\partial \eta \Gamma_0[B]|_{\eta=0}}{\partial \eta \Gamma[B]|_{\eta=0}} \bigg|_{m_{q,i}=0} \Rightarrow \bar{g}^2(L) = g_0^2 + O(g_0^4) \]

- b.c.’s induce a constant colour electric field:

\[ G_{0k} = \partial_0 B_k = \frac{C_k - C'_k}{L} \]

⇒ The coupling is defined as “response coefficient” to a variation of a constant colour electric field.
Example: renormalisation of $P^a = \bar{\psi} \gamma_5 \frac{\tau^a}{2} \psi$:

- In this case we set $C_k = C'_k = 0$, i.e. trivial background field $B = 0$.
- Define correlation functions

$$f_P(x_0) = -\frac{1}{3} \langle O^a P^a(x) \rangle, \quad f_1 = -\frac{1}{3L^6} \langle O^a O'^a \rangle$$
Renormalisation of operators in the SF scheme (2)

- Renormalised correlation functions:
  \[ f_{P,R}(x_0) = Z_\zeta^2 Z_P f_P(x_0), \quad f_{1,R} = Z_\zeta^4 f_1, \]

set \( T = L, \ m = 0, \ x_0 = L/2, \) and impose

\[ Z_P(g_0, L/a) \frac{f_P(L/2)}{\sqrt{f_1}} = \frac{f_P(L/2)}{\sqrt{f_1}} \bigg|_{g_0=0} \]

- similarity with MOM schemes: the renormalised amplitude at \( \mu = L^{-1} \) equals its tree-level expression

- The ratio is formed to cancel any \( Z_\zeta. \)

- definition of running quark mass: \( \overline{m}(L) = Z_P^{-1}(L)m. \)
The aim is to construct the Step Scaling Functions $\sigma(u)$ and $\sigma_P(u)$:

$$\sigma(u) = \bar{g}^2(2L) \big|_{u=\bar{g}^2(L)},$$

$$\sigma_P(u) = \lim_{a \to 0} \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)} \bigg|_{u=\bar{g}^2(L)}$$

These are related to the usual RG functions:

$$\int_{\sqrt{\sigma(u)} \beta(g)}^{\sqrt{u}} \frac{dg}{\sqrt{\sigma(u)} \beta(g)} = \ln 2 \quad \sigma_P(u) = \exp \int_{\sqrt{\sigma(u)} \beta(g)}^{\sqrt{u}} \frac{\tau(g)}{\beta(g)} dg$$

One thus considers a change of scale by a finite factor $s = 2$; RG functions tell us what happens for infinitesimal scale changes.
Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

- choose $g_0$ and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of $u$)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

- now choose $L/a = 6$ and tune $g'_0$ such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- and so on ...
The SSF in the continuum limit

[ALPHA coll., Della Morte et al '05 ]

![Graph showing the SSF behavior with fits to different orders in a plot with the y-axis labeled as \( \sigma(u)/u \) and the x-axis labeled as \( u \).]

- SF scheme, \( N_f=2 \)
- 2-loop
- 3-loop
- non-pert. fit
The running of the SF coupling

\[ [\text{ALPHA coll., Della Morte et al '05} ] \alpha(\mu) \]

\[ \text{SF scheme, } N_f=2 \]

\[ 3\text{-loop} \]

\[ \mu / \Lambda \]

- 250 MeV
- 100 GeV

\[ \text{SF scheme, } N_f=0 \]

\[ 2/3\text{-loop } \beta \]
The formula

\[ \Lambda = \mu \left( b_0 \bar{g}^2 \right)^{-b_1/2b_0^2} \exp \left\{ -\frac{1}{2b_0 \bar{g}^2} \right\} \]

\[ \times \exp \left\{ -\int_{0}^{\bar{g}} \! dx \left[ \frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\} \]

holds for any value of \( \mu \). We may use it at \( L_{\text{min}} \) to obtain

\[ \Lambda L_{\text{min}} = f \left( \bar{g} \left( L_{\text{min}} \right) \right) \]

The function \( f(g) \) can be evaluated at \( g = \bar{g} \left( L_{\text{min}} \right) \) deep in the perturbative region. The integral in the exponent

\[ \int_{0}^{\bar{g}} \! dx \left[ \frac{b_2 b_0 - b_1^2}{b_0^3} x + O(x^3) \right] = \frac{b_2 b_0 - b_1^2}{2 b_0^3} \bar{g}^2 + O(\bar{g}^4) \]

may thus be evaluated using the \( \beta \)-function at 3-loop order.

Since \( L_{\text{max}} = 2^n L_{\text{min}} \) one knows \( L_{\text{max}} \Lambda \)

still need \( F_\pi L_{\text{max}} \)
Matching to a low energy scale

Ideally one would like to compute e.g. $F_\pi \Lambda$, and take $F_\pi = 132\text{MeV}$ from experiment

- What is required? The scale $L_{\text{max}}$ is implicitly defined:

  $$\bar{g}^2(L_{\text{max}}) = 4.84 \Rightarrow (L_{\text{max}}/a)(g_0)$$

  Setting $L_{\text{max}}/a = 6, 8, 10, \ldots$ one then finds corresponding values of the bare coupling (at fixed $g_0$ some interpolation of $L_{\text{max}}/a$ will be necessary instead)

- One must then be able to compute $aF_\pi$ in a large volume simulation at the very same values of the bare coupling:

  $$L_{\text{max}}F_\pi = \lim_{g_0 \to 0} (L_{\text{max}}/a)(g_0)(aF_\pi)(g_0)$$

- One thus needs a range of $g_0$ where both can be computed, $aF_\pi$ and $\bar{g}(L_{\text{max}})$

- Remark: intermediate results are often quoted in terms of Sommer’s scale $r_0$, rather than $F_\pi$. 

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Non-perturbative Renormalization of Lattice QCD
[Alpha collab. ’12 ] Extrapolate the kaon decay constant times $L_1$ to the continuum (analogous $F_\pi L_{\text{max}}$)
The scale \( r_0 \) [R. Sommer '93] is obtained from the force \( F(r) \) between static quark and antiquark separated by a distance \( r \):

\[
r_0^2 F(r_0) = 1.65
\]

The r.h.s. was chosen so that phenomenological estimates from potential models yield \( r_0 = 0.5 \) fm.

Recent result for \( N_f = 2 \) ([ALPHA '12]): \( F_K = 155 \) MeV implies \( r_0 = 0.503(10) \) fm (at physical pion mass!).

Results for \( \Lambda \) using \( r_0 = 0.5 \) fm [ALPHA '99-'12]

\[
\Lambda_{\text{MS}}^{(2)} r_0 = 0.789(52), \quad \Lambda_{\text{MS}}^{(2)} = 310(20) \text{ MeV}
\]

\[
\Lambda_{\text{MS}}^{(0)} r_0 = 0.602(48), \quad \Lambda_{\text{MS}}^{(0)} = 238(19) \text{ MeV}
\]
The running quark mass

- Coupled evolution of the running mass and the coupling:

$$m(2L) = \sigma_m(u)m(L), \quad \sigma_m(u) = 1/\sigma_P$$
$$g^2(2L) = \sigma(u)$$

- Once the running coupling is known in a range $[u_0, u_n]$, determine $\sigma_m(u)$ for the same range of couplings: evolution of quark mass and coupling recursively

$$m(2^k L_{\text{min}})/m(2^{k-1} L_{\text{min}}) = \sigma_m(u_k), \quad k = 1, 2, \ldots, n$$

- one obtains $m(2L_{\text{max}})/m(L_{\text{min}})$
- Extract $m(L_{\text{min}})/M$ using PT as for $\Lambda$-parameter
Running mass in the SF scheme \[ \text{ALPHA '05} \]
In practice with Wilson type quarks, one avoids the additive renormalisation of the bare quark mass parameter by replacing it by a measured bare mass $m_{\text{PCAC}}$ from the (bare) PCAC relation:

$$m_{\text{PCAC}} \stackrel{\text{def}}{=} \frac{\langle \partial_\mu A_\mu^a(x)O \rangle}{2\langle P^a(x)O \rangle}$$

The running quark mass is then related to $m_{\text{PCAC}}$

$$\bar{m}(L) = \frac{1}{Z_P^{-1}(g_0, L/a)Z_A(g_0)} m_{\text{PCAC}}(g_0),$$

where $Z_P^{-1}$ and $Z_A$ are known factors and $m_{\text{PCAC}}(g_0)$ is a measured quantity.

Combine results,

$$M = Z_M(g_0) m_{\text{PCAC}}(g_0)$$

and take the continuum limit $g_0 \to 0$. 
Strange quark mass

The most recent $N_f = 2$ result for the strange quark mass using this strategy ([ALPHA ’12 ]):

$$M_s = 138(3)(1) \text{ MeV} \quad \Rightarrow \quad \overline{m}^{\text{MS}}(\mu = 2 \text{ GeV}) = 102(3)(1) \text{ MeV}$$

- Quoted errors are statistical and systematic;
- **Note**: Except for quenching of the strange quark ALL systematic errors have been addressed!
Concluding remarks

- The recursive finite volume technology has completely eliminated the problem with large scale differences. The RG running is determined in the continuum limit and universal (i.e. regularisation independent)

- To obtain physical results one needs to perform a matching calculation at a low energy scale: it is crucial to have a range in bare couplings where both, the renormalisation conditions and the hadronic input can be computed

- Whether perturbation theory for the running coupling/operator is working well or not down to low scales is not so important; you would not know this beforehand! What error estimate would you have given?!

- Many operator renormalisation problems have been treated already; the technique can be generalised to operators containing static quarks (cf. R. Sommer’s Nara lectures).
Running of the $B_K$ four-quark operator in SF scheme

Quenched approximation [ALPHA collab. '05]

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Non-perturbative Renormalization of Lattice QCD
Continuum vs. lattice symmetries

On the lattice symmetries are typically reduced with respect to the continuum. Examples are

1. Space-Time symmetries: the Euclidean $O(4)$ rotations are reduced to the $O(4,\mathbb{Z})$ group of the hypercubic lattice. Other lattice geometries are possible, even random lattices have been tried.

2. Supersymmetry: only partially realisable on the lattice

3. Chiral and Flavour symmetries:
   - staggered quarks: only a $U(1) \times U(1)$ symmetry remains
   - Wilson quarks: an exact $SU(N_f)_V$
   - twisted mass Wilson quarks: various $U(1)$ symmetries (both axial and vector)
   - overlap/Neuberger quarks: complete continuum symmetries!
   - Domain Wall quarks: (negligibly ?) small violations of axial symmetries; consequences are analysed like for Wilson quarks

Case study: chiral and flavour symmetries with Wilson type quarks
Exact lattice Ward identities (1)

Euclidean action \( S = S_f + S_g \):

\[
S_f = a^4 \sum_x \bar{\psi}(x) \left( D_W + m_0 \right) \psi(x), \quad S_g = \frac{1}{g_0^2} \sum_{\mu, \nu} \text{tr} \{ 1 - P_{\mu\nu}(x) \}
\]

\[
D_W = \frac{1}{2} \left\{ (\nabla_\mu + \nabla^*_\mu) \gamma_\mu - a \nabla^*_\mu \nabla_\mu \right\}
\]

Isospin transformations (\( N_f = 2, \tau^{1,2,3} \) Pauli matrices):

\[
\psi(x) \rightarrow \psi'(x) = \exp \left( i \theta(x) \frac{1}{2} \tau^a \right) \psi(x) \approx (1 + \delta^a_V(\theta)) \psi(x),
\]

\[
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) \exp \left( -i \theta(x) \frac{1}{2} \tau^a \right) \psi(x) \approx (1 + \delta^a_V(\theta)) \bar{\psi}(x)
\]

Perform change of variables in the functional integral and expand in \( \theta \)

\[
\langle O[\psi, \bar{\psi}, U] \rangle = Z^{-1} \int D[\psi, \bar{\psi}]D[U] e^{-S} O[\psi, \bar{\psi}, U].
\]

Due to \( D[\psi, \bar{\psi}] = D[\psi', \bar{\psi}'] \) one finds the vector Ward identity

\[
\langle \delta^a_V(\theta) O \rangle = \langle O \delta^a_V(\theta) S \rangle
\]
Variation of the action, Noether current:

\[
\delta^a_V(\theta)S = -ia^4 \sum_x \theta(x) \partial^*_\mu \tilde{V}^a_{\mu}(x)
\]

\[
\tilde{V}^a_{\mu}(x) = \bar{\psi}(x)(\gamma_\mu - 1)\frac{\tau^a}{4} U(x, \mu) \psi(x + a\hat{\mu})
\]

\[
+\bar{\psi}(x + a\hat{\mu})(\gamma_\mu + 1)\frac{\tau^a}{4} U(x, \mu)^\dagger \psi(x)
\]

Choose region \(R\) and \(\theta\):

\[
R = \{ x : t_1 < x_0 \leq t_2 \}, \quad \theta(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{otherwise} \end{cases}
\]
if $O = O_{\text{ext}}$ is localised outside $R$:

$$0 = \langle O_{\text{ext}} i \delta^a_V(\theta) S \rangle = a^4 \sum_{x_0 = t_1 + a}^{t_2} \sum_x \langle O_{\text{ext}} \partial^*_\mu \tilde{V}_\mu^a(x) \rangle$$

$$= a \sum_{x_0 = t_1 + a}^{t_2} \partial^*_0 \langle O_{\text{ext}} Q^a_V(x_0) \rangle$$

$$= \langle O_{\text{ext}} Q^a_V(t_2) \rangle - \langle O_{\text{ext}} Q^a_V(t_1) \rangle$$

i.e. the vector charge is time-independent;

This expresses the exact vector symmetry on the lattice;

N.B.: These are exact identities between lattice correlation functions!
Choosing $O = O_{\text{ext}} \tilde{V}^{b}_{\mu}(y)$, with $y \in R$:

\[
i \varepsilon^{abc} \left< O_{\text{ext}} \tilde{V}^{c}_{k}(y) \right> = \left< O_{\text{ext}} \tilde{V}^{b}_{k}(y) [Q^{a}_{V}(t_2) - Q^{a}_{V}(t_1)] \right>
\]

\[
i \varepsilon^{abc} \left< O_{\text{ext}} Q^{c}_{V}(y_0) \right> = \left< O_{\text{ext}} Q^{b}_{V}(y_0) [Q^{a}_{V}(t_2) - Q^{a}_{V}(t_1)] \right>
\]

- N.B. The RHS does not vanish since the time ordering matters: $t_1 < y_0$ and $t_2 > y_0$
- Constitutes Euclidean version of charge algebra!
Exact lattice Ward identities (5)

- implies that the Noether current $\tilde{V}_\mu^a$ is protected against renormalisation; if we admit a renormalisation constant $Z_{\tilde{V}}$ it follows that $Z_{\tilde{V}}^2 = Z_{\tilde{V}}$ hence $Z_{\tilde{V}} = 1$; its anomalous dimension vanishes!

- Any other definition of a lattice current, e.g. the local current

$$V_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x), \quad (V_R)_\mu^a = Z_{\tilde{V}}V_\mu^a$$

can be renormalised by comparing with the conserved current. Its anomalous dimension must vanish, i.e.

$$Z_{\tilde{V}} = Z_{\tilde{V}}(g_0) \xrightarrow{g_0 \to 0} 1 + \sum_{n=1}^{\infty} Z_{\tilde{V}}^{(n)} g_0^{2n}.$$
Continuum chiral WI’s as normalisation conditions

- For chiral symmetry there is no conserved current with Wilson quarks.
- However: expect that chiral symmetry can be restored in the continuum limit!

⇒ [Bochicchio et al ’85]: use continuum chiral Ward identities and impose them as normalisation condition at finite lattice spacing $a$!
Continuum chiral WI’s as normalisation conditions

- Define chiral variations:

\[ \delta^a_A(\theta)\psi(x) = i\gamma_5 \frac{1}{2} \tau^a \theta(x) \psi(x), \quad \delta^a_A(\theta)\overline{\psi}(x) = \overline{\psi}(x)i\gamma_5 \frac{1}{2} \tau^a \theta(x) \]

- Derive formal continuum Ward identities assuming that the functional integral can be treated like an ordinary integral:

\[ \Rightarrow \quad \langle \delta^a_A(\theta)O \rangle = \langle O\delta^a_A(\theta)S \rangle, \]

\[ \delta^a_A(\theta)S = -i \int d^4x \theta(x) \left( \partial_\mu A^a_\mu(x) - 2mP^a(x) \right) \]

\[ A^a_\mu(x) = \overline{\psi}(x)\gamma_\mu \gamma_5 \frac{1}{2} \tau^a \psi(x), \quad P^a(x) = \overline{\psi}(x)\gamma_5 \frac{1}{2} \tau^a \psi(x) \]
Simplest chiral WI: the PCAC relation (1)

- Shrink the region $R$ to a point $x$:

$$\langle O_{\text{ext}} \delta^a_A(\theta) S \rangle = 0$$

$$\Rightarrow \quad \langle \partial_\mu A^a_\mu(x) O_{\text{ext}} \rangle = 2m \langle P^a(x) O_{\text{ext}} \rangle$$

- The PCAC relation implies that chiral symmetry is restored in the chiral limit.
Simplest chiral WI: the PCAC relation (2)

- Impose PCAC on Wilson quarks at fixed $a$: define a bare PCAC mass:
  \[ m = \frac{\langle \partial_\mu A_\mu^a(x) O_{\text{ext}} \rangle}{\langle P^a(x) O_{\text{ext}} \rangle} \]

- A renormalised quark mass can thus be written in two ways
  \[ m_R = Z_A Z_P^{-1} m = Z_m (m_0 - m_{cr}) \Rightarrow m = Z_m Z_P Z_A^{-1} (m_0 - m_{cr}) \]

  The critical mass can be determined by measuring the bare PCAC mass $m$ as a function of $m_0$ and extra/interpolation to $m = 0$.

- Note: $m$ is only defined up to $O(a)$; any change in $O_{\text{ext}}$ will lead to $O(a)$ differences.
Determination of the critical mass

PCAC quark mass from SF correlation functions:

\[ m = \frac{\partial_0 f_A(x_0)}{2f_P(x_0)} \]

\( 8^3 \times 16 \) lattice, quenched QCD, \( a = 0.1 \text{ fm} \)
More chiral WI’s: axial current normalisation

- At $m = 0$ we can derive the Euclidean current algebra (in finite volume!):

$$i\varepsilon^{abc}\langle O_{\text{ext}} Q^c_V(y_0) \rangle = \langle O_{\text{ext}} Q^b_A(y_0) [Q^a_A(t_2) - Q^a_A(t_1)] \rangle$$

- Imposing this continuum identity on the lattice (at $m = 0$) fixes the normalisation of the axial current:

$$(A_R)_\mu^a = Z_A(g_0) A^a_\mu, \quad Z_A(g_0) \overset{g_0 \to 0}{\sim} 1 + \sum_{n=1}^{\infty} Z^{(n)}_A g_0^{2n}. $$

- Note: When changing the external fields $O_{\text{ext}}$, the result for $Z_A$ will change by terms of $O(a)$.

- The PCAC relation and the charge algebra become operator identities in Minkowski space. Changing $O_{\text{ext}}$ corresponds to looking at different matrix elements of these operator identities. On the lattice these must be equal up to $O(a)$ terms.
Axial current normalisation with Wilson quarks

$Z_A$ in quenched approximation [Lüscher et al. ’96, Leder & S ’10]

$L = 1.436r_0$

Similar results for $N_f = 2, 3$ by ALPHA collab.